Precoding in Multiple-Antenna Broadcast Systems with a Probabilistic Viewpoint

Amin Mobasher and Amir K. Khandani
Coding & Signal Transmission Laboratory (www.cst.uwaterloo.ca),
Dept. of Elec. and Comp. Eng., University of Waterloo, Waterloo, ON, Canada, N2L 3G1,
E-mail: {amin, khandani}@cst.uwaterloo.ca

arXiv:cs/0701058v2 [cs.IT] 16 Apr 2007

***Abstract*—In this paper, we investigate the minimum average transmit energy that can be obtained in multiple antenna broadcast systems with channel inversion technique. The achievable gain can be significantly higher than the conventional gains that are mentioned in methods like [1]. In order to obtain this gain, we introduce a Selective Mapping (SLM) technique (based on random coding arguments). We propose to implement the SLM method by using nested lattice codes in a trellis precoding framework.**

I. Introduction

Recently, there has been a considerable interest in Multi-Input Multi-Output (MIMO) antenna systems due to achieving a very high capacity as compared to single-antenna systems. Multiuser MIMO systems can also exploit most of the advantages of multiple-antenna systems.

In a broadcast system, when an access point with multiple antennas is used to communicate with many users, the communication is complicated by the fact that each user must decode its signal independently from the others. As a simple precoding scheme, the channel inversion technique can be used at the transmitter to separate the data for different users. However, this method is vulnerable to the poor channel conditions.

In this paper, we investigate the optimum gain for average transmit energy in multiple antenna broadcast systems with channel inversion technique. By using the fact that the channel is not orthogonal, the gain that can be achieved is significantly higher than the regular shaping gains that can be achieved in methods like [1].

In a broadcast system with the channel inversion technique (given a fixed channel matrix), we find the optimal probability distribution for the data vectors to minimize the average transmit energy. Then, we introduce a theoretical Selective Mapping (SLM) technique (based on random coding arguments) to obtain the optimal average transmit energy. In order to implement the SLM method effectively, we propose using nested lattice codes in a trellis precoding framework

The rest of the paper is organized as follows. In Section II, the system model is introduced. Section III finds the optimal probability distribution for transmit data in channel inversion techniques. Section IV is devoted to introducing the SLM technique and its analysis and implementation issues.

[1]This work is supported by the Nortel Networks, the Natural Sciences and Engineering Research Council of Canada (NSERC), and the Ontario Center of Excellence (OCE).

II. System Model

A multiple antenna broadcast system can be modeled by [2]

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \tag{1}$$

where $\mathbf{y}$ is the $\tilde{M}\times 1$ *received vector*, $\mathbf{x}$ is the $\tilde{N}\times 1$ *normalized transmitted data*,$\mathbf{n}$ is additive white Gaussian noise, and $\mathbf{H}$ represents the $\tilde{M}\times\tilde{N}$ channel matrix in real space.

In broadcast systems, the receivers should decode their respective data independently and without any cooperation with each other. The simplest method is using the channel inversion technique as a precoding method at the transmitter to separate the data for different users $\mathbf{s} = \mathbf{H}^{+}\mathbf{u}$, where $\mathbf{H}^{+} = \mathbf{H}^{*}(\mathbf{H}\mathbf{H}^{*})^{-\mathbf{1}}$, $\mathbf{H}^{*}$ is the Hermitian of $\mathbf{H}$, $\mathbf{u}$ is the data vector, i.e. u_i is the data for the *i*'th user, and $\mathbf{s}$ is the transmitted signal before the normalization. When the number of transmit antennas equals with the number of users, $\tilde{M} = \tilde{N} := M$, the transmitted signal is

$$\mathbf{s} = \mathbf{H}^{-\mathbf{1}}\mathbf{u}. \tag{2}$$

As in [1], the normalized transmitted signal would be $\mathbf{x} = \frac{\mathbf{s}}{\sqrt{E\{\gamma\}}}$, where $\gamma = \|\mathbf{s}\|^2$. The problem arises when $\mathbf{H}$ is poorly conditioned and γ becomes very large, resulting in a high power consumption.

In a multiple antenna system, it is assumed that the data vector $\mathbf{u}$ is selected from a constellation with discrete points. However, through this paper we investigate the probabilistic behavior of the transmitted signal $\mathbf{s}$. Assuming a large constellation, continuous approximation provides a probability distribution for each constellation, resulting in different $E\{\gamma\}$. The challenge is finding the best probability distribution with minimum $E\{\gamma\}$. Note that the expectation in $E\{\gamma\}$ is over $\mathbf{u}$ and the channel is assumed constant.

III. Optimum Probabilistic Constellation

Channel inversion technique removes the need for complex decoding algorithms in the receiver side; however, it leads to a high energy consumption as the average energy of the resulting constellation points is high. We are looking for a constellation shaping method for the input constellation, such that using the channel inversion technique, the resulting constellation has a smaller value for the average transmit energy.

A proper input constellation should be designed such that two conditions are satisfied: (i) data can be decoded independently at the receivers, and (ii) the average transmitting energy is as low as possible. The design of the constellation is known as the shaping. By using a conventional block constellation, any point in the constellation is equally likely. However, by shaping, a nonuniform distribution is achieved over each dimension.

A common constellation shaping technique is to choose a finite set of points from an M dimensional lattice $\mathbf{\Lambda}$ that lies within a finite region $\mathcal{R} \subset \mathbb{R}^M$. This constellation is known as a lattice code. If $\mathbb{C}$ is a lattice code of reasonably large size, then the distribution of its points in M dimensional space is well approximated by a uniform continuous distribution over the region $\mathcal{R}$ (the continuous approximation) [3].

Having a uniform distribution over region $\mathcal{R}$ induces a nonuniform distribution on each dimension. In other words, if $\mathbf{u}$ is selected uniformly over $\mathcal{R}$, each element of $\mathbf{u}$ has a nonuniform distribution. Through this paper, the probability distribution of the elements of $\mathbf{u}$ is called *marginal probability distribution* of $\mathbf{u}$. We assume that the region $\mathcal{R}$ has a fixed volume Vol$(\mathcal{R}) = \mathbb{V}$, resulting in the entropy of $(\log \mathbb{V})$. In the case of independent variables, we assume that the entropy per real dimension is $\mathcal{H} = \frac{1}{M} \log \mathbb{V}$.

Let $\mathbf{Q} := \left(\mathbf{H^{-1}}\right)^T \mathbf{H^{-1}} = \mathbf{U\Lambda U^T}$, where $\mathbf{U}$ is the unitary matrix of eigenvectors of $\mathbf{Q}$ and $\mathbf{\Lambda}$ is the diagonal matrix of the corresponding eigenvalues, $\lambda_i, \;\; i = 1, \cdots, M$. Assume $\mathbf{u} \in \mathcal{R}$ be a random vector with mean $E\{\mathbf{u}\} = \boldsymbol{\mu}$ and the correlation matrix $E\{\mathbf{u}\mathbf{u}^T\} = \mathbf{\Sigma} > 0$. The energy of the transmitted signal, called *transmit energy*, is defined as $\gamma = \mathbf{u}^T\mathbf{Q}\mathbf{u} = \mathbf{u}^T\mathbf{U\Lambda U}^T\mathbf{u} = \mathbf{v}^T\mathbf{v}$, where $\mathbf{v} = \sqrt{\mathbf{\Lambda}}\mathbf{U}^T\mathbf{u}$. When $\mathbf{u}$ is selected uniformly in $\mathcal{R}$, the vector $\mathbf{v}$ is selected uniformly over a region $\mathcal{R}'$, where $\mathcal{R}' = \{\mathbf{v}|\mathbf{v} = \sqrt{\mathbf{\Lambda}}\mathbf{U^T}\mathbf{u}, \forall \mathbf{u} \in \mathcal{R}\}$. It is more convenient to explain some behaviors of γ based on $\mathbf{v}$. The average transmit energy can be written as [2]

$$E\{\gamma\} = tr(\mathbf{Q\Sigma}) + \boldsymbol{\mu}^T\mathbf{Q}\boldsymbol{\mu} \tag{3}$$

If we ignore that users are supposed to decode their data independent of each other, the optimum region for the input constellation can be found using the following lemma:

Theorem 1: Let $\mathbf{u} = [u_1, u_2, \cdots, u_M] \in \mathbb{R}^M$ be a random vector with probability distribution $f(u_1, u_2, \cdots, u_M)$, mean $E\{\mathbf{u}\} = \boldsymbol{\mu}$, and the correlation matrix $E\{\mathbf{u}\mathbf{u}^T\} = \mathbf{\Sigma} > 0$, in a broadcast system introduced in (1). Let $\mathcal{H}(\mathbf{u})$ denote the entropy of the data vector $\mathbf{u}$. Then, a multivariate Gaussian random vector $\mathbf{u}$ with $\boldsymbol{\mu} = \mathbf{0}$ and the covariance matrix

$$\mathbf{\Sigma} = \sqrt[M]{\Pi\lambda_i}\sigma^2\mathbf{H}\mathbf{H}^T \tag{4}$$

will minimize the energy of the transmit signal given a fixed entropy $\mathcal{H}(\mathbf{u}) = \log(\mathbb{V})$, where σ^2 is the variance of a Gaussian random variable with entropy $\mathcal{H} = \frac{1}{M}\log(\mathbb{V})$.

Proof: See [2] ■

This choice of $\mathbf{\Sigma}$ suggests that the minimum value of the average energy among transmit signals with different probability distributions is [2]

$$E_{opt} = E\{\gamma\} = M \sqrt[M]{\Pi\lambda_i}\sigma^2 \tag{5}$$

Consider the auxiliary vector $\mathbf{v} = \sqrt{\mathbf{\Lambda}}\mathbf{U}\mathbf{u}$. It can be easily shown that each element of this vector has a Gaussian distribution with zero mean with variance $\mathcal{R}_{eq}^2$. Therefore, in the limit of $M \longrightarrow \infty$, this vector is uniformly selected over an M-dimensional sphere centered at the origin with radius $\sqrt{M}\mathcal{R}_{eq}$, i.e. $\mathcal{B}_M(0, \sqrt{M}\mathcal{R}_{eq})$ (corresponding to the minimum average transmit energy in (5).

Roughly speaking, we can assume that the region $\mathcal{R}'$ is $\mathcal{B}_M(0, \sqrt{M}\mathcal{R}_{eq})$. On the other hand, the vector $\mathbf{u}$, a Gaussian random vector with zero mean and covariance matrix in (4), is uniformly selected over the region $\mathcal{R}$ which is an oval. The main diameters of this oval are along the eigenvectors $\mathbf{U}$ and the radii of the oval in each direction are $\sqrt{\frac{M}{\lambda_i}}\mathcal{R}_{eq}$ for $i = 1, \cdots, M$, in other words $\mathcal{R} = \mathcal{O}_M(0, \sqrt{\frac{M}{\lambda_i}}\mathcal{R}_{eq})$. (see [2]).

By using this region, an additional *channel gain* of [2]

$$\mathcal{G}_{\mathbf{H}} = \frac{Arithmetic\ Mean(\lambda_1, \cdots, \lambda_N)}{Geometric\ Mean(\lambda_1, \cdots, \lambda_N)} \tag{6}$$

can be achieved (in addition to the conventional shaping gain).

The geometric mean of a data set is always smaller than or equal to the set's arithmetic mean (the two means are equal if and only if all members of the data set are equal). On the other hand, without the channel matrix, we have the conventional shaping gain. However, the presence of $\mathbf{H}^{-1}$ will affect the shaping gain by the *Channel Gain*, $\mathcal{G}_{\mathbf{H}}$, defined in (6). Without the channel effect the optimum region $\mathcal{R}$ is a spherical region (corresponding to independent Gaussian variables), while with the channel effect the optimum region $\mathcal{R}$ is an M-dimensional oval.

From another point of view, this gain can be seen as the effect of rate (or power) allocation for Gaussian distribution which has been considered in multi-carrier transmission and point to point multiple antenna systems, e.g. [4]. However, this concept ignores the independency condition required for a broadcast system. Here, the challenging problem is how the region $\mathcal{R}$ or $\mathcal{R}'$ can be achieved, while considering the independency condition.

IV. Selective Mapping

The idea of Selective Mapping (SLM) is to generate a large set of data vectors that represent the same information, where the data vector resulting in the lowest energy is selected for transmission. This idea has been used in OFDM systems, e.g. [5], to reduce the average transmit energy.

In the optimum case, the vector $\mathbf{u}$ is selected uniformly over an M-dimensional oval and the transmit vector is selected uniformly over a hypersphere. However, due to the independency condition, implementing this oval shape region is not possible. The receivers can not co-operate with each other to locate a point inside this oval. We propose an SLM method that can theoretically achieve the optimum gain for average transmit energy. The region for vector $\mathbf{u}$ is not oval; however, the resulting region for the transmit vector in the limit is a hypersphere.

In the sequel, first, we use a random coding argument to explain the SLM method, its analysis, and the maximum

theoretical gain that can be achieved. In this part, again we ignore the independency condition. In continue, we implement the SLM technique considering the independency condition by using a trellis precoding.

In the system model (1), the volume of the region is fixed, $\mathbb{V}$. In order to provide multiple choices for the SLM method, the volume is increased to $\bar{\mathbb{V}}$ such that for each data vector there are N points, where $N = \frac{\bar{\mathbb{V}}}{\mathbb{V}}$.

In other words, N i.i.d. samples of $\mathbf{u}$ are generated, $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_N\}$, and $\mathbf{s}_l$ with the lowest transmit energy is selected for transmission. In other words, $\gamma_l = \min\{\gamma_1, \gamma_2, \cdots, \gamma_N\}$. We are looking for the probabilistic behavior of γ_l.

A. Asymptotic Analysis

In this section, we analysis the effect of an SLM method for broadcast systems. In the proposed method, N i.i.d. samples of $\mathbf{u}$ are generated, $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_N\}$, and among the corresponding transmit vectors $\mathbf{s}_i = \mathbf{H}^{-1}\mathbf{u}_i$, the vector $\mathbf{s}_l$ with the lowest transmit energy is selected for transmission. In other words, in the SLM method, we are looking for

$$\min_{1 \le i \le N} \|\mathbf{s}_i\|^2, \tag{7}$$

where $\|.\|$ represents the regular norm.

The expression in (7) is similar to minimization of distortion in quantization and random quantizers. The tremendous research on random quantization [6, and ref. therein] can help us to evaluate the expression in (7) in our SLM method.

Let $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_N$, be i.i.d. $\mathbb{R}^M$-valued random variables with distribution Q, i.e.

$$Q(\mathbf{v}) = \mathbb{P}\{s_{i_1} \le v_1, \cdots, s_{i_M} \le v_M\} \quad i = 1, \cdots, N, \tag{8}$$

where

$$\mathbf{v} = (v_1, \cdots, v_M) \in \mathbb{R}^M.$$

For any region $\mathcal{R}$, the probability $Q(\mathcal{R})$ is the probability that there is at least one code point in the region $\mathcal{R}$, i.e.

$$Q(\mathcal{R}) = \int_{\mathcal{R}} Q(d\mathbf{y}).$$

Define the r^{th} *order transmit energy* as

$$\gamma_{r,N}^Q = \min_{1 \le i \le N} \|\mathbf{s}_i\|^r, \tag{9}$$

where based on our previous notation $\gamma_l = \gamma_{2,N}^Q$. In this section, the asymptotic probabilistic behavior of $\gamma_{2,N}^Q$, when $N \longrightarrow \infty$, is investigated. Specifically, we calculate the average transmit energy in the SLM technique. Note that, in the following, we frequently use λ which is the M-dimensional Lebesgue measure. Here, we define it as the M-dimensional volume of a region.

Theorem 2: Let $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_N$, be i.i.d. $\mathbb{R}^M$-valued random variables with distribution Q. Then,

$$\lim_{N \to \infty} E\left\{N^{\frac{r}{M}} \gamma_{r,N}^Q\right\} = B_M^{-\frac{r}{M}} \Gamma(1 + \frac{r}{M}) g_\rho^{-\frac{r}{M}} \tag{10}$$

where $B_1 = 2$, $B_M = \lambda\left(\mathcal{B}_M(0,1)\right) = \pi^{M/2}/\Gamma(1 + M/2)$ for $M = 2, \cdots$, and g_ρ is defined for any $\rho > 0$ as

$$g_\rho := \inf_{\delta \in (0,\rho]} \frac{Q\left(\mathcal{B}_M(0,\delta)\right)}{\lambda\left(\mathcal{B}_M(0,\delta)\right)}.$$

Proof: See [2]. ■

Now, consider the special case of uniform distribution. When we have a large lattice code, we can assume we have a uniform distribution over the region where the lattice code is defined. Applying SLM technique, over a region with uniform distribution results in the following average for the r^{th} order transmit energy.

Theorem 3: Let $\mathcal{R} \subset \mathbb{R}^M$ be a compact set with $\lambda(\mathcal{R}) > 0$ and let $\mathbf{s}_1, \cdots, \mathbf{s}_N$ be i.i.d. random variables with uniform distribution over $\mathcal{R}$. Then,

$$\lim_{N \to \infty} E\left\{N^{\frac{r}{M}} \gamma_{r,N}^Q\right\} = B_M^{-\frac{r}{M}} \Gamma(1 + \frac{r}{M}) \lambda(\mathcal{R})^{\frac{r}{M}}. \tag{11}$$

Proof: Let Q be a uniform distribution over $\mathcal{R}$, i.e. $Q = U(\mathcal{R})$. Therefore,

$$Q\left(\mathcal{B}_M(0, \frac{v^{\frac{1}{r}}}{N^{\frac{1}{M}}})\right) = \frac{\lambda\left(\mathcal{B}_M(0, \frac{v^{\frac{1}{r}}}{N^{\frac{1}{M}}})\right)}{\lambda(\mathcal{R})}, \tag{12}$$

and

$$g_\rho = \inf_{\delta \in (0,\rho]} \frac{Q\left(\mathcal{B}_M(0,\delta)\right)}{\lambda\left(\mathcal{B}_M(0,\delta)\right)} = \frac{1}{\lambda(\mathcal{R})}. \tag{13}$$

Substituting (13) in (10) completes the proof. ■

Note that we are interested in cases that the i.i.d random variables $\mathbf{u}_1, \cdots, \mathbf{u}_N$ are selected uniformly over a region $\mathcal{R}'$. According to $\mathbf{s} = \mathbf{H}^{-1}\mathbf{u}$, for the probability distribution of $\mathbf{s}$, we have

$$f_{\mathbf{s}}(\mathbf{s}) = |\mathbf{H}^{-1}| \, f_{\mathbf{u}}(\mathbf{H}\mathbf{s}). \tag{14}$$

Therefore, if $\mathbf{u}$ has a uniform distribution over $\mathcal{R}'$, $\mathbf{s}$ has also a uniform distribution over $\mathcal{R}$, where $\mathcal{R} = \mathbf{H}^{-1}\mathcal{R}'$.

In order to find the asymptotic average transmit energy of SLM technique, we should replace $r = 2$ and $Q = U(\mathcal{R})$, where $\mathcal{R}$ is the region for the transmit vector $\mathbf{s}$. Therefore, according to the expression in (11), the average transmit energy for large N can be approximated by [2]

$$E_{SLM} = \Gamma(1 + \frac{2}{M}) M \mathcal{R}_{eq}^2. \tag{15}$$

Comparing (5) and (15), we can see that using SLM technique with any lattice code of reasonably large size the optimum transmit energy can be achieved since for large M, $\Gamma(1 + \frac{2}{M}) \longrightarrow \Gamma(1) = 1$.

Corollary 1: In a broadcast system, applying SLM method to lattice codes of reasonably large size, with a fixed volume, will result in equal values for the average transmit energy when N is large enough in the SLM method.

We must emphasis that in our random coding argument the probability of the event that two different code words have the same transmit data vector is negligible. In the case of this event, we have an error in our broadcast system. However, since the probability of this event is small, the average transmit energy would not change.

B. Implementation Issues

In any practical SLM method, the lattice code $\mathbb{C}$ (constellation) should be expanded such that the number of constellation points are multiplied by N, resulting in a new lattice code $\mathbb{C}'$. This new set of constellation points are grouped in $|\mathbb{C}|$ sets containing N points. Transmitting any of these N points transfer the same information. These sets (and the expanded constellation) should be selected such that the users at the receive side can decode their data independent of each other.

The method proposed in [1] can be considered as an SLM technique with this idea. In this method the region for the transmit vector $\mathbf{s}$ is expanded by repetition of the constellation by multiples of τ in each direction. In other words, for any vector $\mathbf{s} = \mathbf{H}^{-1}\mathbf{u}$, we find $\mathbf{H}^{-1}(\mathbf{u}+\tau\mathbf{l})$ for $\lfloor -b/2 \rfloor + 1 \leq l_i \leq \lfloor b/2 \rfloor$. In each direction we repeat the constellation b times, so we have $N = b^M$ in the SLM method. In this method, the transmit vector $\mathbf{s}$ is selected in the original constellation and $N-1$ other points are calculated by adding integer vector offsets, resulting in N points in the expanded lattice code $\mathbb{C}'$. A modulo operation in the transmitter and receivers guarantees and satisfies the independency condition.

For large enough M and N, this method can not achieve the optimum average energy. The equivalent region for vector $\mathbf{s}$ is the Voronoi region of $\tau\mathbf{H}^{-1}$, not a hypersphere [2]. This leads to an improvement over the Gaussian marginal probability distribution; however, this is not the best that we can achieve. The more this Voronoi region looks like a sphere, the less the average transmit energy is. The problem in the SLM method in [1] is that the vector $\mathbf{s}$ is not uniformly distributed over lattice code $\mathbb{C}'$. In order to preserve the independency condition, a vector is uniformly distributed over $\mathbb{C}$ and $N-1$ other points are calculated deterministically in $\mathbb{C}'$ based on this point. This results in a region with Voronoi region shape not a sphere.

In [7], a sign-bit shaping algorithm is proposed for precoding in broadcast systems. Sign-bit shaping is implemented by using a trellis code. This technique is actually an SLM method since it gives the transmitter many different options when determining which symbol to transmit.

Trellis shaping systems are composed of a rate (k_s, n_s) binary convolutional shaping code C and a signal set A partitioned into 2^{n_s} shaping subsets [8]. The signal set A is typically a lattice code with shaping region $\mathcal{R}$, and the shaping subsets are the points of this region that fall within subregions $\mathcal{R}_i$ for $i = 1, \cdots, 2^{n_s}$. It is important that the code C and the signal set A is selected such that the the equivalent points are selected uniformly over $\mathcal{R}$.

Conventional precoding schemes in broadcast systems, such as [1], treat multiple antennas of different users as different users. By using trellis shaping for each virtual user, in each 2-dimensional space, there is a modulo operation with respect to the Voronoi region of the shaping trellis code. In other words, in the space of each user, there is a modulo operation with respect to the Cartesian product of these Voronoi Regions. However, we can use the shaping concept in each user's space. The idea in [7] can be extended to include the multi-antenna case. We can use a trellis shaping for each user, and not for each antenna.

Nested lattice codes can be implemented such that both these improvements are met. The idea of nested lattice codes has already been used for interference cancelation in degenerated broadcast systems [9]. There, it is assumed that, in an ordered set of users, each user has the ability that it can decode the message for the previous users. In other words, it is assumed that each user has the code-book for the previous users. This technique can be implemented in our scheme to provide us with shaping, without any need for these assumptions. We can achieve the same gain as that reported in [9] for broadcast systems with precoding, without any extra assumption.

Assume that in a broadcast system with K users, each user has n_u antenna in (1), i.e. $M = 2Kn_u$. One way of implementing this idea is implementing a large trellis consisting of K sub-trellises with a lattice partition, $\mathbf{\Lambda}/\mathbf{\Lambda}'$, in a $2n_u$ dimensional space. In each sub-trellis, the lattice code is divided into $|\mathbf{\Lambda}/\mathbf{\Lambda}'|$ partitions, i.e. for transmitting any information vector, one of $|\mathbf{\Lambda}/\mathbf{\Lambda}'|^K$ equivalent points, in M dimensional space, with the lowest transmit energy is selected. In other words, the vector $\mathbf{u}$ resulting in the lowest energy for $\mathbf{H}^{-1}\mathbf{u}$ would be selected for transmission. Now, in each $2n_u$ dimensional space, the modulo operation is with respect to the Voronoi region of this trellis code. The Cartesian product of these regions should be as close as possible to the sphere in order to generate the optimum shaping region.

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